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DEPARTMENT OF MATHEMATICS

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1. If α, β, γ are linearly independent vectors of $V(R)$, then show that $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$ are also linearly independent.

Sol: Consider the scalars $a,\ b,\ c \in R$ such that

$$a(\alpha+\beta)+b(\beta+\gamma)+c(\gamma+\alpha)=0$$

$$\Rightarrow (a+c)\alpha+(a+b)\beta+(b+c)\gamma=0$$

Given that α, β, γ are also linearly independent. This implies that

$$a+c=0,\ a+b=0,\ b+c=0.$$

By solving the above equations, we get $a=0,\ \ b=0,\ c=0$ and therefore the given vectors $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are also linearly independent.

Theorem: Let $V(F)$ be a vector space and $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a finite subset of non-zero vectors of $V(F)$. Then S is linearly dependent if and only if some vector $\alpha_k \in S, 2 \leq k \leq n,$ can be expressed as a linear combination of its preceding vectors.

Proof: Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a finite subset of non-zero vectors of the vector space $V(F)$.

Case 1: Let S be linearly dependent. Then there exist scalars $a_1, a_2, \dots, a_n \in F$, not all zeros such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_n\alpha_n = \bar{0} - - - - - - (1)$$

Let k be the smallest suffix such that $a_k \neq 0$.

From (1), $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_k\alpha_k + 0\alpha_{k+1} + 0\alpha_{k+2} + \cdots + 0\alpha_n = \bar{0}$.ss

Suppose $k = 1$, then $a_1\alpha_1 = \bar{0}$. But $\alpha_1 \neq \bar{0}$, then $a_1 = 0$. This is contradiction to $a_i's$ are not all zeros and therefore $2 \leq k \leq n$.

Again from (1), $a_k\alpha_k = -a_1\alpha_1 - a_2\alpha_2 - \cdots - a_{k-1}\alpha_{k-1}$

$$\Rightarrow \alpha_k = (-a_k{}^{-1}a_1)\alpha_1 + (-a_k{}^{-1}a_2)\alpha_2 + \cdots + (-a_k{}^{-1}a_{k-1})\alpha_{k-1}$$

This shows that α_k can be written as the linear combination of its preceding vectors, for $2 \le k \le n$.

Case 2: Let α_k can be written as the linear combination of its preceding vectors, for $2 \le k \le n$, then

$$\alpha_k = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{k-1}\alpha_{k-1}, \text{ where } b_1, b_2, \dots, b_{k-1} \in F.$$

$$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{k-1}\alpha_{k-1} + (-1)\alpha_k = \bar{0}, \text{ here } -1 \neq 0.$$

Therefore the set of vectors $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ are linearly dependent and hence the set of vectors $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ are linearly dependent.

THANK YOU